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SURVEY ON THE OPTIMAL CONTROL

OF LANCHESTER-TYPE ATTRITION PROCESSES

James G. Taylor

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This is a survey paper on the author's research on applications of the mathematical theory of optimal control/differential games to problems of military conflict in order to study the structure of optimal tactical allocation policies. This program has been carried out for tactical allocation problems with the combat described by Lanchester-type equations of warfare. Both deterministic and stochastic attrition processes have been considered, although the major emphasis has been on the former. Optimal allocation policies have been developed for numerous one-sided optimization problems of tactical

interest with deterministic Lanchester-type attrition processes in order to study the dependence of the structure of these optimal policies upon model form. Problem areas in applying current mathematical theories to solve such problems are discussed. An important gap in the existing theory of differential games is identified. Various attrition models are considered (reflecting different assumptions as to target acquisition process, command and control capabilities, target engagement process, variations in range capabilities of weapon systems).

P R E F A C E

This report surveys previous research on combining dynamical optimization and combat modelling theories in order to gain insights into the optimization of combat dynamics. This material was presented by the author at the Symposium on the State-of-the-Art of Mathematics in Combat Models held at McLean, Virginia (General Research Corporation) on 14-15 June 1973.

As in any survey, details have been suppressed for the sake of perspective. Further details are to be found, however, in the author's previous technical report (reference 35) and, of course, in the open literature.

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March 1, 1974

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1. Introduction.

In this paper we will review the results of the author's studies of some idealized models for the optimization of combat dynamics and their possible implications for defense planners. Our research approach has been to combine Lanchester-type formulations of combat attrition (both deterministic and stochastic) and generalized control theory* [15] (both deterministic and stochastic optimal control, dynamic programming, differential games). In this review we will summarize our past results. Details are to be found in [35] and in the open literature.

A quantitative theory of tactical allocation has been developed through the examination of a sequence of simplified** models. These combat models are far too simple to be taken literally but should be interpreted as indicating general principles to serve as hypotheses for subsequent higher resolution studies via computer simulation or field experimentation. The effects of various modelling assumptions upon the structure of the optimal allocation policies have been systematically studied by contrasting the solutions for various models (see [36]).

A major result of our research has been that optimal tactical allocation policies are quite sensitive to the precise type of model adopted, even as to whether the tactical scenario lasts for a specified period of time or terminates only when a pre-determined system state has been reached. Insights have been provided into such important questions as:

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*This term was apparently first coined by Y. C. Ho in [14].

**However, the complete solution of these dynamical optimization problems is at the frontier of applications of generalized control theory to operations research.

(1) How should fire be distributed over targets?

(2) How should targets be selected?

(3) Do target priorities change over time?

(4) Do force levels affect the optimal allocation policy?

(5) How does the number of target types affect the optimal allocation policy?

(6) Do conflict termination circumstances affect the optimal allocation policy?

(7) How is the optimal fire distribution/target selection policy affected by the nature of the attrition process?

(8) What is the effect of logistics constraints on such policies?

(9) How does the uncertainty and confusion of combat affect optimal allocation policies?

To develop our theory of tactical allocation we have extensively studied some specific combat scenarios. Optimal tactics for the following have been studied: selection of target type at which to fire, regulation of firing rate. The influences of the following factors have all been considered:

(1) combatant objectives (form of criterion function and valuation of surviving forces),

(2) weapon system performance characteristics,

(3) termination conditions of conflict,

(4) force strengths and composition,

(5) type of attrition process,

(6) effects of resource (logistics) constraints,

(7) range capabilities of weapon systems.

The tactical situations that have been studied are described by deterministic Lanchester-type equations of warfare. The combat continues over a period of time with a choice of tactics available to both sides and subject to change over time. The mathematical theory of deterministic optimal control/differential games has been used to solve the problems under consideration.

In 1964 Dolansky [8] noted that the Lanchester theory of combat was insufficiently developed in the area of target selection for combat between heterogeneous forces (optimal control/differential games). Even the two references cited by him, Weiss [47] and Isbell and Marlow [18], have been subsequently extended by the author [32], [38]. Since Dolansky's article, the only work known to this author (except for that contained in Isaacs' book [17]) has been by Chattopadhyay [4], [5], Etter [9], Intriligator [16], Kawara [20], Moglewer and Payne [22], and Sternberg [30]. A further review of this work and a discussion of its relationship to that of the author are to be found in [35].

2. Elements of the Combat Optimization Problem.

One may consider that there are three essential parts of any dynamical combat optimization problem:

(1) the decision criteria (for both combatants),

(2) the model of conflict termination conditions (and/or unit breakpoints),

(3) the model of combat dynamics.

In the opinion of this author our knowledge of the topics in the above list increases as we go down the list. Relatively little is known about such decision making criteria and even less has appeared in the literature.*

*Since this conference was held, the following noteworthy article has appeared in the open literature [26].

Some work on models of conflict termination has appeared [12], [48], [49], although its real impact has not yet apparently been felt. Although our knowledge is still far from perfect in modelling combat dynamics, an extensive literature does exist (see [41] or [11]).

In the course of our research we have become increasingly aware of the importance of the first two elements above, and we feel that more basic research is needed on them. Other items that could be added to the above list are: (a) the information structure and (b) the actual decision latitude of the combatants. It should be pointed out that in essentially all (see Section 3.5 of [36] for an exception) the problems that we have considered, it has been assumed that each decision maker has perfect information about the state variables (e.g. force levels) and model parameters.

3. Overview of Research Program.

In developing our quantitative theory of tactical allocation we have considered the following subject areas:

(1) combat attrition models,

(2) deterministic versus stochastic models of combat attrition,

(3) special features of deterministic optimal control models for tactical allocation.

Thus, in performing our research we have tried to bring together several bodies of knowledge which previously existed as well-defined entities but without any interaction. A detailed discussion of these topics and their interactions is to be found in [35].

The reader will note that our research has concentrated primarily on one-sided dynamical optimization problems. This has been done for two reasons: (1) the well-known intimate connection between optimal control theory and differential games (see Section 4 of [36]), and (2) the lack of certain key results in the current theory of differential games (see below). Accordingly, our research approach has been to study optimal fire distribution problems for dynamic combat situations with decisions available to both sides by first considering corresponding one-sided (optimal control) problems. After the difficult points have been mastered in the simpler one-sided problems, then the differential games are most profitably tackled.

4. Lanchester-Type Differential Games.

In a two-person zero-sum deterministic differential game with closed-loop (or feedback) strategies [15] (henceforth simply referred to as a differential game) each player chooses a strategy (for precise definitions see [1] or [10]) in order to maximize his own criterion functional (which when added to that of his opponent yields zero) for a system whose dynamics are described by a system of ordinary differential equations. Without being precise, it seems appropriate to refer to a differential game as being a Lanchester-type differential game when the system dynamics are described by Lanchester-type equations of warfare.

It should be pointed out that contemporary research on applications of differential games has concentrated primarily on pursuit-evasion problems (see the survey papers by Berkovitz [2], Y. C. Ho [15], and Simakova (USSR literature) [28]) or, more recently, economics problems. Theoretical research has focused on questions of existence of (a) value of the differential game [6], [7], [10] and (b) a saddle point in pure strategies [10]. As we shall see, Lanchester-type differential games contain certain more or less unique aspects (most important of which is the presence of state variable inequality constraints) that have not been addressed adequately in this previous research.

An example of a fairly general Lanchester-type differential game is as follows:

$$\underset{\Psi_{ij}}{\text{maximize}}\ \underset{\Phi_{ij}}{\text{minimize}} \left\{ \sum_{i=1}^{n} w_i y_i(t_f) - \sum_{i=1}^{m} v_i x_i(t_f) \right\},$$

$$\text{stopping rule:} \quad t_f - T = 0,$$

$$\begin{array}{ll} \text{subject to:} & \dfrac{dx_i}{dt} = r_i - \sum_{j=1}^{n} \psi_{ij} a_{ij} y_j \quad \text{for } i = 1,\ldots,m, \\ \text{(combat dynamics)} & \\ & \dfrac{dy_i}{dt} = s_i - \sum_{j=1}^{m} \phi_{ij} b_{ij} x_j \quad \text{for } i = 1,\ldots,n, \end{array}$$

with

$$x_i, y_j \geq 0 \quad \text{for all } i,j \quad \text{(State Variable Inequality Constraints)}$$

$$\left.\begin{array}{l} \sum_{i=1}^{m} \psi_{ij} \leq 1 \quad \text{for } j = 1,\ldots,n, \\ \sum_{i=1}^{n} \phi_{ij} \leq 1 \quad \text{for } j = 1,\ldots,m, \\ \phi_{ij}, \psi_{ij} \geq 0, \end{array}\right\} \text{(Strategic Variable Inequality Constraints)}$$

where

$x_i(t)$, $y_i(t)$ are force levels,

r_i, s_i are replacement rates,

v_i, w_i are the utilities assigned survivors,

a_{ij} is the rate at which one Y_j unit can destroy X_i,

b_{ij} is the rate at which one X_j unit can destroy Y_i,

ϕ_{ij} is the fraction of X_j who fire at Y_i,

and ψ_{ij} is the fraction of Y_j who fire at X_i.

The above problem (1) contains all the essential features (and can be made identical to by minor changes in formulation) of all the Lanchester-type differential games studied previously in the operations research literature (see Section 1 above).

First and foremost to the mathematician is the question as to whether a differential game such as (1) has value. The significance of this question for the operations analyst is that its answer tells him whether he has chosen an appropriate formulation for his problem. Considering results of Friedman (see Chapter 6 of [10]), it may be shown* that such two-sided fire distribution problems in the Lanchester theory of combat do have value and a saddle point in pure strategies (see [33], [36], [39]). An essential requirement which leads to this result is that the Hamiltonian be separable, i.e. a function independent of ϕ plus a function independent of ψ.** It should be pointed out that this need not be true for other dynamical structures. For example, when defensive capabilities were considered in the attrition process in a tactical air war game extensively studied at RAND, the resulting model did not possess a solution in pure strategies (see [36]).

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*Provided that certain technical requirements are met. Most remarkable is the fact that the existence of value for such a differential game cannot be guaranteed unless replacements are allowed.

**We use the symbol ϕ to denote all the strategic variables under the control of X.

It is the opinion of the author that the current-state-of-the-art for solving differential games is not sufficient to allow routine solution of a problem like (1). This statement is made in full knowledge of numerous papers (including some of the author's) that have appeared in the literature. Problem areas in the mathematical theory of Lanchester-type differential games then are as follows:

(1) no adequate theory of state variable inequality constraints (SVIC's) currently exists,*

(2) synthesis of extremals,

(3) determination of optimal strategies from extremal strategic-variable pairs,

(4) presence of singular surfaces in the solution.

A detailed discussion of the above is to be found in [35]. For now, let us just touch the high points.

One of our research findings [35] has been that a theory of state variable inequality constraints** (which is essential for solving problems such as (1) because of the requirement that force levels be non-negative (see also [43])) is lacking in the current theories for differential games.*** In fact, the treatment in the literature of SVIC's in one-sided

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*Since the conference, the author has succeeded in developing necessary conditions of optimality for differential games with SVIC's [46]. Applications of these results are also given in this report.

**In (1) x_i for $i = 1,\dots,m$ and y_j for $j = 1,\dots,n$ are state variables, while the ϕ's and ψ's are called strategic variables. An example of a SVIC is the condition that $x_1 \geq 0$, i.e. the X_1 force level must be non-negative.

***A. Friedman (see Chapter 6 in [10]) and others have discussed the existence of value for differential games with SVIC's. However, the analogues of the well-known control theory multiplier conditions for constrained subarcs [19] are not developed. In fact, the treatment of the example on pp. 239-240 of [10] by Friedman is inadequate, since his approach fails to yield optimal strategies for minor modifications in the problem's formulation [46].

optimal control problems has not been entirely satisfactory: in studying the problems discussed in this paper, the author has made some contributions on SVIC's to the optimal control theory literature [34], [42].

The synthesis of extremals from the basic necessary conditions of optimality can be complicated even in a one-sided version of (1) (see [36]). The determination of optimal strategies from extremal strategic-variable pairs can be quite complex even in the case of the simplest terminal control differential game (called game of survival by Friedman [10]) (see [39]). It is well-known (see [13], [17]) that singular surfaces (in the sense of Isaacs (see pp. 132-134 of [17])) play a key role in solving differential games. The author has frequently encountered (in Isaacs' terminology) universal surfaces and dispersal surfaces in Lanchester-type differential games of interest. We will consider one-sided problems below with singular subarcs (i.e. their solution contains a universal surface in Isaacs' terminology) and dispersal surfaces in their solution.

Because of the above noted difficulties and the lack of theoretical results for SVIC's, the author attempted to master such difficulties by first considering simpler one-sided dynamic combat optimization problems. It then turned out that the state-of-the-art of deterministic optimal control theory was not sufficient to allow <u>routine</u> solution of even the simplest Lanchester-type optimal control problem [33], [43].

5. Special Features of Deterministic Optimal Control Models for Tactical Allocation.

Before we consider examples of Lanchester-type optimal control problems, we will discuss some special features of such problems which make solving them (even today) a non-routine matter. These remarks are based on the author's experiences in trying to solve numerous particular problems. The special features of deterministic Lanchester-type optimal control problems are then as follows:

(1) non-uniqueness of extremals,

(2) state variable inequality constraints,

(3) singular subarcs.

Again, we will summarize the main points here with a detailed discussion to be found in [35].

The purpose of our research has been to determine optimal fire distribution/target selection policies for dynamic tactical allocation problems. However, as the reader is undoubtedly aware, the maximum principle (or the max-min principle for differential games) only provides necessary conditions of optimality. It is customary to refer to a path on which the necessary conditions of optimality (maximum principle) are satisfied everywhere in time* as an extremal. Thus, one must demonstrate the optimality of an extremal trajectory. Two ways in which this may be done are as follows:

(a) check whether sufficient conditions of optimality are satisfied on the extremal,

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*This means that the necessary conditions of optimality have been developed for the class of piecewise continuous controls.

(b)* by citing the appropriate existence theorem,** show that an optimal control exists for the problem under consideration; there are two further subcases: (1) if the extremal is unique, then it is optimal or (2) if the extremal is not unique and only a finite number exist, then the optimal trajectory is determined by considering a finite number of alternatives.

It has not been convenient to take the former approach (see [38], [43], [45]). (It is noteworthy that for problems with "square-law" attrition of target types (see [36], [37]) (like problem (2) below) the problem may be cast in a form for which all the sufficient conditions of (global) optimality are satisfied <u>except</u> the requirement that the planning horizon be of fixed length.***)

Thus, we have taken the second approach to demonstrating the optimality of an extremal. If an extremal is unique, then it is optimal and no difficulty exists. However, in the simplest possible fire distribution problem we have shown that extremals are not unique for a certain range of model parameters [33], [43]. In other words, local extrema exist for these problems and each one must be examined. One may even have to contend with a <u>dispersal surface</u> (see pp. 132-141 of [17]) being present in the solution.**** This rare singular surface is difficult to determine, since its presence cannot be determined directly from the maximum principle

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*This is essentially a condensation of the general solution procedure given in [24], [33], and [39] (see also Appendix G of [35]).

**Unfortunately, existence theorems are usually proven (see [21]) for the class of measurable control functions so that a gap exists between the classes of functions for which the necessary conditions of optimality are developed and those for which existence theorems are proven.

***This is because such a battle is terminated anytime a side is annihilated.

****This occurs in both the supporting weapon system game of H. K. Weiss [39] and the Isbell-Marlow fire distribution problem [33].

but requires "considerations in the large." The existence of multiple extremals in Lanchester-type optimal control problems has important implications for devising computational methods.

In all Lanchester-type dynamic tactical allocation problems the force levels are required to be non-negative* (or an equivalent condition). In such models, force levels will be represented by state variables so that such a restriction is mathematically called a state variable inequality constraint (SVIC). The maximum principle (in its original form [23]) is inadequate to handle SVIC's, and separate necessary conditions of optimality** apply to subarcs which lie on the boundary of the state space. When we began our research, we found that results were widely scattered in the literature and that a completely adequate theory to solve even the simplest fire distribution problem did not exist. Moreover, our subsequent research on the "simplest" fire distribution problem has led to several contributions on SVIC's to the control theory literature [34], [42]. It finally should be noted that lack of a theory of SVIC's is a gap in the current theory of differential games.

As we have seen above, the strategic variables (e.g. fraction of X_j who fire at Y_i) appear linearly in Lanchester-type differential games (see equation (1) above). In the control theory literature (Isaacs [17] uses different terminology) such problems are called singular (because the matrix of second partial derivatives of the Hamiltonian with respect to the strategic variables is a singular matrix) and require special treatment. One-sided versions of such fire distribution problems then are

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*Negative force levels do not make physical sense.

**See [19] for an almost comprehensive review of results for optimal control problems with SVIC's.

singular problems of optimal control. For such problems the maximum principle may not determine the singular control and the classical theory does not provide adequate tests of optimality for singular subarcs (see [38]).

A mathematical theory of singular (in the control theory sense) subarcs is essential for solving tactical allocation problems within the Lanchester theory of combat because it is required to identify optimal fire distribution policies that are not extreme points of the control variable space (i.e. policies that are other than 0 or 1). The author has shown that an optimal fire distribution policy, denoted as ϕ^*, such that $0 < \phi^* < 1$ arises in a simple problem of fire distribution for a homogeneous force in Lanchester combat against two enemy force types each of which undergoes attrition at a rate proportional to the product of the number of firers and targets [38]. Finally, there may be other types of singular surfaces (in the sense of Isaacs (see pp. 132-134 of [17])) in the solution to Lanchester-type optimal control problems of interest.

6. The Isbell-Marlow Fire Programming Problem.

The Isbell-Marlow fire programming problem [18] is probably the simplest optimal control problem that arises in the Lanchester theory of combat. Consequently, the development of a complete solution to this problem along with the appropriate solution methodology is vital for guiding extensions to situations of more complex combat dynamics (both one-sided and two-sided). The author views this problem then as a "benchmark" case to which the treatment (both theoretical and computational) of more complicated problems can (and should) be related.* Moreover, several useful

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*For example, one could test the capability of a computational approach like Lagrange dynamic programming [25] on a discrete-time version of this problem.

insights into the structure of optimal fire distribution policies in more general cases may be obtained from this simpler problem.

The simplest fire distribution problem is for combat between a heterogeneous X force of two force types (for example, riflemen and grenadiers) and a homogeneous Y force (for example, riflemen only). This situation is shown diagramatically in Figure 1. Let us consider the case in which it is the objective of the Y-force commander to maximize the net worth of survivors (considering a linear utility for the number of each force type) at the end of battle. This is accomplished through his fraction of fire, denoted as ϕ, directed at X_1. We will consider a battle lasting until one side or the other is totally annihilated* and will discuss more general "breakpoints" below.

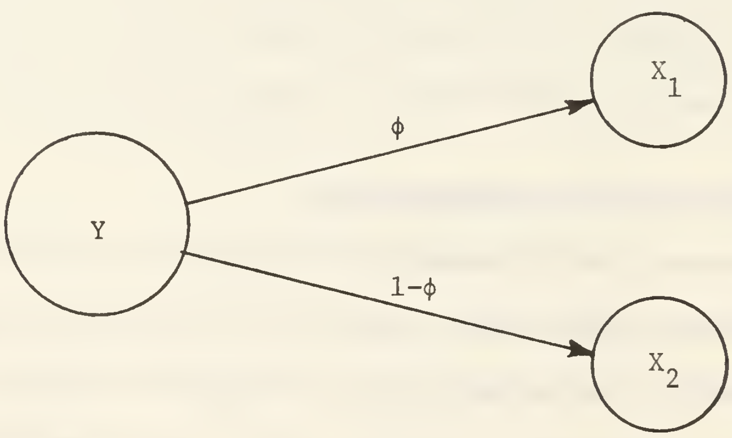

Figure 1. Diagram of Fire Programming Problem.

*This is a special case of a terminal control battle in which the end of battle is determined by the system being steered to a specified end point.

Mathematically the problem may be stated as

$$\underset{\phi(t)}{\text{maximize}} \{ry(T) - px_1(T) - qx_2(T)\},$$

subject to: (combat dynamics)

$$\frac{dx_1}{dt} = -\phi a_1 y,$$

$$\frac{dx_2}{dt} = -(1-\phi) a_2 y, \tag{2}$$

$$\frac{dy}{dt} = -b_1 x_1 - b_2 x_2,$$

with initial conditions

$$x_1(t=0) = x_1^o, \qquad x_2(t=0) = x_2^o, \qquad y(t=0) = y_0,$$

and

$0 \le \phi \le 1$ (Control Variable Inequality Constraints),

$x_1, x_2, y \ge 0$ (State Variable Inequality Constraints),

with the stopping rule

(a) $y(T) = 0,$

or (b) $x_1(T) = x_2(T) = 0,$ (3)

where

p, q, and r are linear utilities assigned to surviving force types,

x_1, x_2, and y are force levels,

a_1, a_2, b_1, and b_2 are constant attrition-rate coefficients,

and ϕ is the fraction of the Y force who fire at X_1.

The optimal fire distribution policy for the above problem (2), (3) is given in [33] (with some further refinements given in [43]).* Because

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*It is expressed as an open-loop control, i.e. $\phi^* = \phi^*(t;t_0,x_i^o,y_0)$, in [33] and [43] (see [14]).

of the complexity of the complete solution, it will not be given here. However, we will summarize the essential characteristics of the optimal fire distribution policy, denoted as ϕ^*. These are as follows:

(1) ϕ^* is always 0 or 1 (except for at most one point in time),

(2) parameters on which the optimal policy depends are

(a) whether Y wins or loses,

(b) $R = a_1b_1/(a_2b_2)$,

(c) $\delta = a_1p/(a_2q)$.

There are some important interpretations of the above: (A) a_ib_i is a measure of strategic value of firing at X_i (rate of destruction of X_i's kill capability against Y), and (B) a_1p is measure of short-run return to Y from firing at X_1 at end of battle (rate of destruction of X_1 value at end of battle).

The most significant aspect of the optimal fire distribution policy for (2) and (3) is that it depends directly upon the force levels alone.* This result is remarkable because the maximum principle does not directly involve the state variables (i.e. the force levels) when the Hamiltonian is maximized for $x_1, x_2 > 0$. This above statement revises remarks made in [36]. Furthermore, the optimal policy for Y may be different depending on whether he wins or loses. Assuming that $R = a_1b_1/(a_2b_2) > 1$, then if Y is going to win, $\phi^* = 1$ for $x_1 > 0$. If Y is going to lose, then the solution depends on another parameter ($\delta = a_1p/(a_2q)$) and is very complicated to express as a closed-loop policy. However, it is instructive to discuss the general features of Y's optimal fire distribution policy

*In other words, we can express the optimal policy as a closed-loop control $\phi^* = k(x_1,x_2,y)$, and the force level dependence is real.

when Y loses. Let $p = k(1+\gamma)b_1$ and $q = kb_2$, where γ is a parameter which reflects whether the surviving X_1 forces are valued more $(\gamma>0)$ or less $(\gamma<0)$ than in direct proportion to their kill capability. Then $p = q(b_1/b_2)(1+\gamma)$, and it is readily shown that $\gamma = -1 + \delta/R$. Moreover, we have (1) $\gamma = 0$ means that surviving enemy weapon systems are valued in direct proportion to their kill capability; (2) for $\gamma \geq -(1-1/R)$, simple solution: $\phi^* = 1$ for $x_1 > 0$; (3) for $-(1-1/R) > \gamma \geq -\sqrt{1-1/R}$, it is complicated to obtain solution;* and (4) for $-\sqrt{1-1/R} > \gamma \geq -1$, it is <u>very</u> complicated to obtain solution. In the latter two cases, ϕ^* is 1 and then changes to 0 later. When this change occures is the complicated part.

The above has an important implication: in battle a commander must use his judgment to ascertain to what end the course of battle can be steered so that he may devise his strategy accordingly. Computationally, it means that to solve such a problem one must know to which extremal end states** the system can be steered (i.e. what force levels are required to drive the system to a target set such that appropriate necessary conditions of optimality are satisfied at the end). The author has discovered that the above characteristic is present in <u>all</u> Lanchester-type optimization problems. In other words, it turns out that considerations "in the large" dominate obtaining the optimal policy over time in such a dynamic optimization problem.

Let us illustrate one of the computational difficulties (multiple extremals) in obtaining the optimal policy discussed above. In Table I

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*It should be noted that for $R > 1$ we have $0 < 1 - 1/R < 1$.

**The author has developed theoretical results along this line.

is shown the results of applying the maximum principle* in the first phase (see [33]) of solving (2) and (3). The following are definitions of parameters appearing in this table:

$$s = b_1x_1^o + b_2x_2^o,$$

$$A = [z^2(R-1)-R]/(z-1)^2,$$

$$B = A(z-1)^2/z^2 = [z^2(R-1)-R]/z^2.$$

The table has been developed by working backwards from each extremal end state of battle. In Table I, if the initial force levels $P^o = (x_1^o, x_2^o, y_0)$ are such that P^o belongs to the domain of controllability (see [33]) for the terminal state C_1 (denoted as $D(C_1)$), then Y can steer the course of battle to this end state with the (open-loop) extremal control shown in the table. It turns out that several of the domains of controllability shown in Table I overlap so that for a given set of initial force levels there may be more than one candidate optimal course of battle. One can compare the returns from these alternatives (a finite number). This is how the optimal (open-loop) fire distribution policy shown in Table II has been obtained from the information of Table I. An outline of the details of this determination is as follows [43].

Let $D(C_i)$ denote the domain of controllability for extremals leading to C_i and P_i denote the return associated with an extremal leading to C_i. For example, it may be shown that

$$P_1 = \left(\frac{-q}{b_2R}\right)\sqrt{R}\,\sqrt{s^2 + (R-1)(b_2x_2^o)^2 - a_1b_1y_0^2}.$$

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*A subtle theoretical result required to obtain these results is that sign restrictions on certain terminal multipliers depend on whether or not replacements are allowed (i.e. in a problem like (9) whether or not $r_1 > 0$) [42].

Table I. Extremals for Isbell-Marlow Problem for

$R - \sqrt{R(R-1)} < \delta < 1.$

Nonrestrictive Assumption: $R > 1$, i.e. $a_1b_1 > a_2b_2$

Case (2): $R - \sqrt{R(R-1)} < \delta < 1$ where $\delta = a_1p/(a_2q)$

Terminal State	Extremal Control	Domain of Controllability
$C_1 \begin{cases} x_1(t_1) = 0 \\ x_2(T) > 0 \\ y(T) = 0 \end{cases}$	$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_1 \\ 0 & \text{for } t_1 < t \le T \end{cases}$	$a_1b_1y_0^2 < s^2 + (R-1)(b_2x_2^0)^2$ $a_1b_1y_0^2 > s^2 - (b_2x_2^0)^2$
$C_2 \begin{cases} x_1(t_1) = 0 \\ x_2(T) = 0 \\ y(T) > 0 \end{cases}$	$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_1 \\ 0 & \text{for } t_1 < t \le T \end{cases}$	$a_1b_1y_0^2 > s^2 + (R-1)(b_2x_2^0)^2$
$C_4 \begin{cases} x_1(t_2) > 0 \\ x_2(T) = 0 \\ y(T) = 0 \end{cases}$	$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_2 \\ 0 & \text{for } t_2 < t \le T \end{cases}$	$a_1b_1y_0^2 \ge R\{s^2 - (b_1x_1^0)^2\}$ $a_1b_1y_0^2 \le s^2 + A(b_2x_2^0)^2$
$C_5 \begin{cases} x_1(T) > 0 \\ x_2(T) > 0 \\ y(T) = 0 \end{cases}$	$\phi^*(t) = 0$ for $0 \le t \le T$	$a_1b_1y_0^2 \le Rs^2\{1 - 1/z^2\}$ $a_1b_1y_0^2 < R\{s^2 - (b_1x_1^0)^2\}$
$C_5^S \begin{cases} x_1(T) > 0 \\ x_2(T) > 0 \\ y(T) = 0 \end{cases}$	$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le T-\tau_1 \\ 0 & \text{for } T-\tau_1 < t \le T \end{cases}$	$a_1b_1y_0^2 > Rs^2\{1 - 1/z^2\}$ $a_1b_1y_0^2 > s^2 + A(b_2x_2^0)^2$ $a_1b_1y_0^2 < s^2 + B(b_2x_2^0)^2$

Definition of Times

(a) t_1 is first t such that $x_1(t_1) = 0$.

(b) t_2 is first t such that $2b_1x_1(t_2)x_2^o + b_2(x_2^o)^2 = a_2y^2(t_2)$.

(c) τ_1 is determined by $\cosh \sqrt{a_2b_2}\,\tau_1 = \dfrac{R - \delta}{R - 1}$.

Table II. Solution to Isbell-Marlow Problem for

$R - \sqrt{R(R-1)} < \delta < 1.$

Nonrestrictive Assumption: $R > 1$, i.e. $a_1b_1 > a_2b_2$

Case (2): $R - \sqrt{R(R-1)} < \delta < 1$ where $\delta = a_1p/(a_2q)$

Terminal State	Optimal Control	Region of Initial Force Levels
$C_1 \begin{cases} x_1(t_1) = 0 \\ x_2(T) > 0 \\ y(T) = 0 \end{cases}$	$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_1 \\ 0 & \text{for } t_1 < t \le T \end{cases}$	$a_1b_1y_0^2 < s^2 + (R-1)(b_2x_2^0)^2$ $a_1b_1y_0^2 \ge s^2 + B(b_2x_2^0)^2$
$C_2 \begin{cases} x_1(t_1) = 0 \\ x_2(T) = 0 \\ y(T) > 0 \end{cases}$	$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_1 \\ 0 & \text{for } t_1 < t \le T \end{cases}$	$a_1b_1y_0^2 > s^2 + (R-1)(b_2x_2^0)^2$
$C_4 \begin{cases} x_1(t_2) > 0 \\ x_2(T) = 0 \\ y(T) = 0 \end{cases}$	$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_2 \\ 0 & \text{for } t_2 < t \le T \end{cases}$	$a_1b_1y_0^2 \ge R\{s^2 - (b_1x_1^0)^2\}$ $a_1b_1y_0 \le s^2 + A(b_2x_2^0)$
$C_5 \begin{cases} x_1(T) > 0 \\ x_2(T) > 0 \\ y(T) = 0 \end{cases}$	$\phi^*(t) = 0$ for $0 \le t \le T$	$a_1b_1y_0 \le Rs^2\{1 - 1/z^2\}$ $a_1b_1y_0 < R\{s^2 - (b_1x_1^0)^2\}$
$C_5^S \begin{cases} x_1(T) > 0 \\ x_2(T) > 0 \\ y(T) = 0 \end{cases}$	$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le T-\tau_1 \\ 0 & \text{for } T-\tau_1 < t \le T \end{cases}$	$a_1b_1y_0^2 > Rs^2\{1 - 1/z^2\}$ $a_1b_1y_0^2 > s^2 + A(b_2x_2^0)^2$ $a_1b_1y_0^2 < s^2 + B(b_2x_2^0)^2$

Definitions of Times: for t_1, t_2, and τ_1, see Table I.

Then, by direct computation of the return functional (considerations "in the large") we have [43]

(1) $P_4(P^o) > P_1(P^o)$ for all $P^o \in \{D(C_1) \cap D(C_4)\}$,

(2) $P_5(P^o) > P_1(P^o)$ for all $P^o \in \{D(C_1) \cap D(C_5)\}$,

(3) $P_5^S(P^o) > P_1(P^o)$ for all $P^o \in \{D(C_1) \cap D(C_5^S)\}$.

It should be noted that $D(C_4) \cap D(C_5) = \Phi$, $D(C_4) \cap D(C_5^S) = \Phi$, and $D(C_5) \cap D(C_5^S) = \Phi$, where Φ denotes the empty set.

Also of interest is whether the nature of the scenario (i.e. planning horizon) affects the structure of the optimal fire distribution policy. Accordingly, we can consider a "prescribed duration" version of the problem (2) in which the battle can last a maximum time of T_1 (which is specified). The stopping rule for this "prescribed duration" battle is then

(a) $x_1(T) = x_2(T) = 0$ and $T \leq T_1$,

(b) $y(T) = 0$ and $T \leq T_1$,

or (c) $T = T_1$, (4)

where T denotes the time at which the battle ends. Again (see [24] or Appendix G of [35]), ϕ^* turns out to be 0 or 1 except for at most one point in time, although the dependence of ϕ^* on t, x_1, x_2, and y is even more complicated than for the terminal control battle (2), (3). There is also an additional parameter $\left(\alpha = \frac{r}{q}\sqrt{\frac{b_2}{a_2}}\right)$ upon which the solution may depend. The most significant result, though, is that the optimal policy may not be the same as for the previous case (see Appendix G of [35]). Hence, even the scenario may affect the optimal policy.

Finally, it seems appropriate to discuss the extension of the above model (2), (3) to more "realistic" breakpoints* (the force level at which a unit ceases to be effective). From historical sources, it is clear that the stopping rule (3) is not in consonance with empirical evidence. One's initial reaction to this state of affairs might be to retain the combat attrition equations (2) but modify (3) to more commonly accepted breakpoint levels. For example, it is frequently assumed that an attacking unit "breaks" at seventy per cent of its initial strength. The author has come to the conclusion [44] that <u>if such a breakpoint is considered, the combat attrition equations must also be modified</u>. For illustrative purposes, let us consider combat between two homogeneous forces, denoted as X and Y.

Assuming that fire is aimed and target acquisition is not a constraining factor, one might hypothesize the following familiar Lanchester-type equations as describing the combat

$$\frac{dx}{dt} = -ay,$$

$$\frac{dy}{dt} = -bx. \tag{5}$$

If such a breakpoint hypothesis is combined with equations (5), there is a rather sharp discontinuity in unit effectiveness when a unit reaches its breakpoint [44]. Considering work by Spring and Miller at RAND [29], the author has hypothesized the following Lanchester-type equations [44] (see also [27])

$$\frac{dx}{dt} = -a\left\{1 - \left(\frac{y_0 - y}{y_0 - y_{BP}}\right)^n\right\} y,$$

$$\frac{dy}{dt} = -b\left\{1 - \left(\frac{x_0 - x}{x_0 - x_{BP}}\right)^m\right\} x, \tag{6}$$

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*See [12] for a criticism of breakpoint hypotheses.

where x_{BP} denotes the force level at which the X unit becomes ineffective (i.e. its breakpoint). If heterogeneous-force versions of (6) are used in place of (2), then the resulting optimal control problem is not amenable at all to analytic solution. The model's mathematical tractability has been sacrificed to "realism." However, many aspects of the development of the solution to the original problem (2), (3) are still applicable and provide insights for guiding the computational solution to the more complicated version.

7. "Linear-Law" Attrition of Target Types.

It is convenient to refer to an attrition process in which the attrition rate of a target type is proportional only to the number of firers as a "square-law" attrition process. Similarly, it is convenient to refer to an attrition process in which the attrition rate of a target type is proportional to the product of the numbers of firers and targets as a "linear-law" attrition process.* As pointed out in [36], a "square-law" attrition process is characterized by constant returns over time per unit of weapon system allocated, whereas a "linear-law" attrition process is characterized by diminishing returns. These different properties lead to an essential difference in the structures of the optimal fire distribution policies.

Let us therefore, consider a version of (2) in which each of the X-force target types undergoes a "linear-law" attrition process in a prescribed duration battle.

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*The circumstances under which these attrition processes have been hypothesized to arise are reviewed in [40].

$$\underset{\phi(t)}{\text{maximize}}\ \{ry(T) - px_1(T) - qx_2(T)\} \quad \text{with } T_1 \text{ specified,}$$

$$\text{subject to:} \quad \frac{dx_1}{dt} = -\phi a_1 x_1 y,$$

$$\frac{dx_2}{dt} = -(1-\phi) a_2 x_2 y,$$

$$\frac{dy}{dt} = -b_1 x_1 - b_2 x_2, \tag{7}$$

$$0 \leq \phi \leq 1, \quad x_1, x_2, y \geq 0, \quad \text{and} \quad T \leq T_1.$$

The solution to (7) was developed in [38] and contrasted to that of (2), (3) and other models in [36].

There is a fundamental difference between the structure of the optimal fire distribution policy for (2), (3) and that for (7): when target types undergo a "linear-law" attrition process, the optimal distribution of fire does not have to be an extreme point in the control variable space at all times. In other words, ϕ^* may be other than 0 or 1. The theory of singular extremals (see [38]) is required to solve this problem with ϕ^* such that $0 < \phi^* = a_2/(a_1+a_2) < 1$ being the "singular" control. Again, the optimal policy depends directly upon force levels (and possibly time). In [38] we showed that for constant attrition-rate coefficients no change occurs in the ranking of target priorities when survivors are assigned a linear utility in direct proportion to their kill rate against Y, and this is independent of whether target types undergo a "square-law" or a "linear-law" attrition process.

The solution to (7) for $\frac{p}{q} = \frac{b_1}{b_2}$ is shown in Figure 2. When $a_1 b_1 x_1 > a_2 b_2 x_2$, the optimal policy is to concentrate all fire on X_1. The line $a_1 b_1 x_1 = a_2 b_2 x_2$ (denoted as L) is a singular "surface" and

Case (a) $\frac{p}{q} = \frac{b_1}{b_2}$

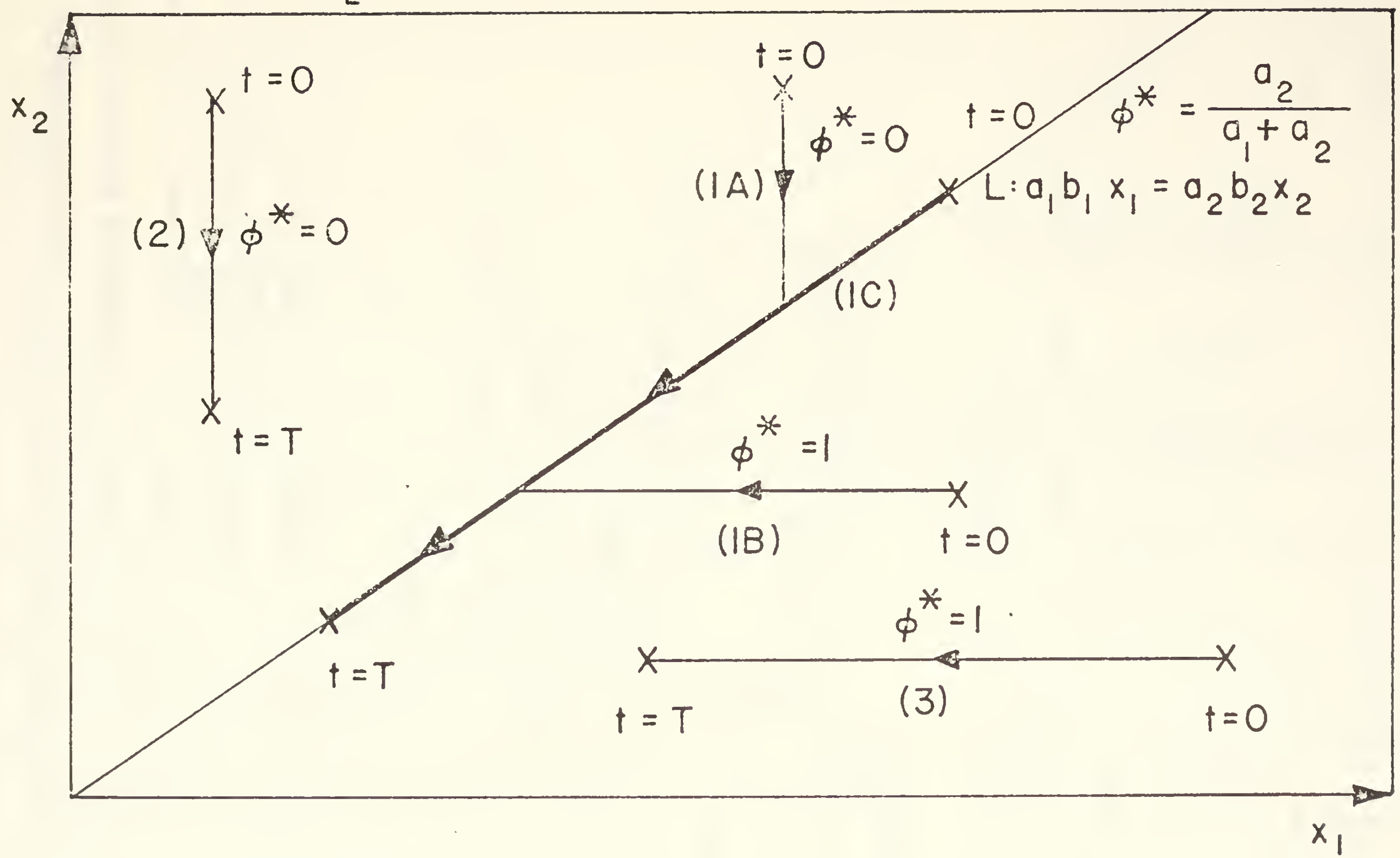

Figure 2. Optimal allocation for linear-law attrition process–
Survivors valued in direct proportion to kill rates.

divides the state space into decision regions. When a force level trajectory reaches L, fire is split over the target types such that the trajectory stays on L. For $\frac{p}{q} > \frac{b_1}{b_2}$ the solution is shown in Figure 3. The battle is divided into two time phases: Phase I for $0 \le t \le t_I$ and Phase II for $t_I < t \le T$. During Phase I the optimal target engagement policy at a point in time is determined by the location of the point on the battle trajectory with respect to the line L, which is the singular "surface." During Phase II the optimal target engagement policy is to use $\phi^* = 1$ below L' (with equation $a_1 p x_1 = a_2 q x_2$) and $\phi^* = 0$ above L'.

8. Further Extensions.

Further extensions of the above basic problems that have been studied by the author are as follows:

(1) replacements [37],

(2) n-versus-one combat [36] (see also Appendix E in [35]),

(3) variable attrition-rate coefficients [37],

(4) other combat attrition forms (see Appendix K in [35]),

(5) combat attrition modeled as a stochastic process [24],

(6) command and control aspects [45],

(7) logistics aspects [3].

Our research findings on many of the above are summarized on pp. 59-64 of [35]. It should be pointed out that our work on the two basic problems (2) and (7) has been essential in guiding these extensions. We will now give formulations for some of the above and occasionally make some remarks.

In [37] we considered the following problem with time-dependent attrition-rate coefficients and replacements.

Case (b) $\frac{p}{q} > \frac{b_1}{b_2}$

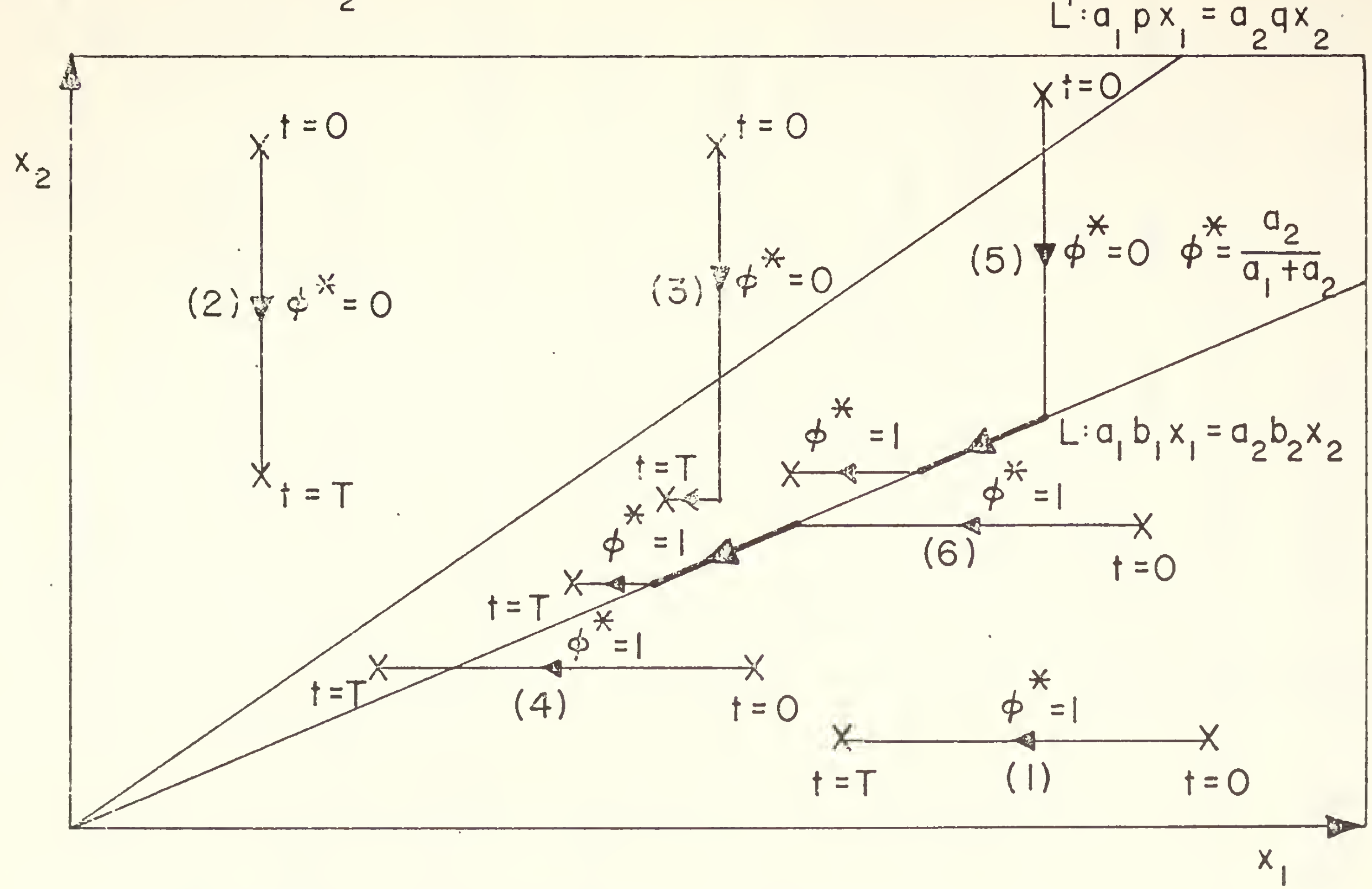

Figure 3. Optimal allocation for linear-law attrition process – Survivors not valued in direct proportion to kill rates.

$$\underset{\phi(t)}{\text{maximize}} \{ry(T) - px_1(T) - qx_2(T)\} \quad \text{with} \quad T_1 \quad \text{specified},$$

$$\text{subject to:} \quad \frac{dx_1}{dt} = -\phi a_1(t)y + r_1(t),$$

$$\frac{dx_2}{dt} = -(1-\phi)a_2(t)y + r_2(t),$$

$$\frac{dy}{dt} = -b_1(t)x_1 - b_2(t)x_2 + s(t), \tag{8}$$

$$0 \leq \phi \leq 1, \quad x_1, x_2, y \geq 0, \quad \text{and} \quad T \leq T_1.$$

In this paper we gave the first application of the theory of SVIC's to allocation problems in the Lanchester theory of combat. A rather remarkable theoretical result that is required for the solution of such a problem is the following: the multiplier corresponding, for example, to the terminal inequality constraint $x_1(T) \geq 0$ is restricted in sign only if $r_1(T) > 0$ [42]. Some of our results in [37] are based on our work on analytic solutions to variable coefficient Lanchester-type equations [32], [40].

We have also generalized these results to n-versus-one combat [37] (see also Appendix E of [35]).

$$\underset{\phi_i(t)}{\text{maximize}} \{vy(T) - \sum_{i=1}^{n} w_i x_i(T)\} \quad \text{with} \quad T_1 \quad \text{specified},$$

$$\text{subject to:} \quad \frac{dx_i}{dt} = -\phi_i a_i y + r_i \quad \text{for} \quad i = 1, \ldots, n,$$

$$\frac{dy}{dt} = -\sum_{i=1}^{n} b_i x_i, \tag{9}$$

$$\sum_{i=1}^{n} \phi_i \leq 1, \quad \phi_i \geq 0, \quad x_i, y \geq 0, \quad \text{and} \quad T \leq T_1.$$

A rather illuminating result is that is $w_i = kb_i$ for $i = 1,\ldots,n$, then the optimal fire distribution policy for Y (at least when $y(T) > 0$) is to always concentrate all his fire on the available target type with largest $a_i b_i$.

More recently we have examined the effects of command and control limitations (in the sense that there is a limitation on how fast fire can be redistributed) on such fire distribution policies [45].

$$\max_{u(t)} \{ry(T) - px_1(T) - qx_2(T)\} \quad \text{with } T_1 \text{ specified,}$$

$$\text{subject to:} \quad \frac{dx_1}{dt} = -\phi a_1 y,$$

$$\frac{dx_2}{dt} = -(1-\phi)a_2 y,$$

$$\frac{dy}{dt} = -b_1 x_1 - b_2 x_2,$$

$$\frac{d\phi}{dt} = u, \tag{10}$$

$$0 \le \phi \le 1, \qquad x_1, x_2, y \ge 0, \qquad -R_L \le u \le R_U, \quad \text{and} \quad T \le T_1.$$

It turns out [45] that such command and control limitations do not essentially alter the optimal fire distribution decision rules, although the shifting of fires is initiated earlier when command and control limitations exist than when an entire force can instantaneously shift their fires from one target type to another.

Finally, we have considered the optimal control of Lanchester-type stochastic processes. For example, we have studied the following problem [24].

$$\underset{\phi}{\text{maximize}}\ E[rN(t_f) - pM_1(t_f) - qM_2(t_f)] \quad \text{with} \quad t_f \quad \text{specified},$$

Subject to: casualties occur randomly as a continuous parameter Markov chain with stationary transition probabilities corresponding to deterministic "square-law" attrition process (2), (11)

$$M_1, M_2, N \geq 0 \quad \text{and} \quad 0 \leq \phi \leq 1,$$

where the random variables $M_1(t)$, $M_2(t)$, and $N(t)$ are force levels (integers) and $E[\cdot]$ denotes mathematical expectation. It is shown in [24] that the structure of the optimal fire distribution policy remains essentially the same when casualties occur randomly, although the optimal policy followed by Y in a realization of the stochastic combat process may differ appreciably from that for the deterministic formulation if the realization does not "follow" the deterministic trajectory.

Further discussion of the above problems and further results are to be found in [35].

9. Model Which can be Extended to Justify General Patton's Tactics in Europe in World War II.

In this section we will describe a simple model which, nevertheless, can generate some insights into tactics used by General George S. Patton, Jr., in World War II. Consider combat between X and Y forces. Part of the X forces (denoted as X_2) can be kept in reserve and consequently do not consume supplies as rapidly as the combatant forces (denoted as X_1) do. Due to "pipeline" capacity there is an upper limit to the rate at which supplies can be replenished. The decision variable under the control of the X commander is the rate of reinforcing (or withdrawing for $u(t)$ negative) the X_1 forces. This situation is shown in Figure 4.

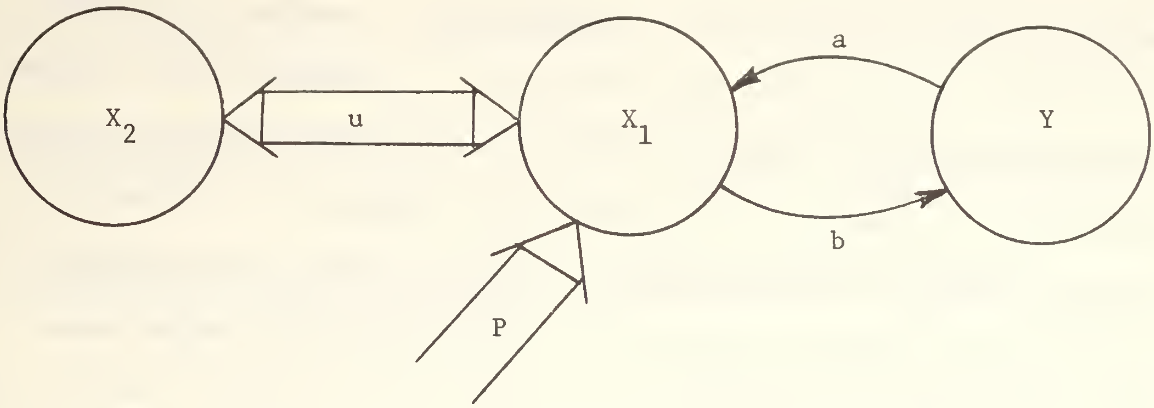

Figure 4. Diagram of Supply-Constrained Combat Problem.

In mathematical terms, the problem is as follows.

$$\underset{u(t)}{\text{maximize}} \{px_1(T) + qx_2(T) - ry(T)\} \quad \text{with} \quad T \quad \text{unspecified},$$

$$\text{subject to:} \quad \frac{dx_1}{dt} = -ay + u,$$

$$\frac{dx_2}{dt} = -u,$$

$$\frac{dy}{dt} = -bx_1,$$

$$\frac{dS}{dt} = -cx_1 + P,$$

$$-W \le u \le R \quad \text{and} \quad x_1, x_2, y, S \ge 0,$$

with the stopping rule

(a) $x_1(T) = 0$,

(b) $y(T) = 0$,

or (c) $S < 0$,

where P denotes the "pipeline" capacity and S denotes the supply level of the X_1 forces. Currently, one of my Ph.D. students in Operations

Research at USNPGS (LCDR Robert L. Powers, USN) is studying this problem (see [3] for a discussion of its formulation). If supplies are to become a constraining factor for the X forces later in combat, then the optimal tactic is to "overcommit" forces early in the campaign (i.e. forces are being withdrawn at the moment when the supply constraint becomes active). Of particular mathematical difficulty is the presence of a second order SVIC in this problem (see [42]).

10. Optimization of Combat Dynamics: Conclusions.

Some conclusions that we have reached from our study of the optimization of combat dynamics are as follows:

(1) The nature of the attrition process has a significant effect upon optimal strategies.

(2) Force levels do affect optimal strategies. Whether one is going to "win" (superiority) or "lose" (inferiority) affects what one's optimal strategy will be.

(3) Even the nature of the scenario (e.g. terminal control or prescribed duration conflict) may affect optimal strategies.

(4) Optimal tactics are also significantly influenced by the nature of the target acquisition process and command and control capabilities.

A further discussion is to be found in [35].

It finally should be remarked that in all our studies above we have noticed that optimal tactics are intimately related to the end to which the combat can be steered. Thus, one must first find out where he can go before he worries about optimizing. The author feels that future research should center upon looking for planning horizon theorems (which divide a conflict into "phases"). The type of question that is to be asked is: under what circumstances are optimal tactics in one phase independent of other combat phases?

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